ADVANCES IN APPLIED MATHEMATICS **10**, 164–174 (1989)

Tuscan-K Squares*

TUVI ETZION, SOLOMON W. GOLOMB, AND HERBERT TAYLOR

University of Southern California, Los Angeles, California 90089-0272

We prove that an $n \times (n+1)$ polygonal path circular Tuscan-n array exists only if $n+1$ is prime. Two results proved in [1] are illustrated. One says that an $n \times n$ polygonal path Tuscan-$(n-1)$ square exists if and only if an $\infty \times n$ singly periodic Costas array exists. The other gives a construction for K orthogonal $(n+1) \times (n+1)$ Latin squares, if an $n \times (n+1)$ circular Tuscan-K array exists. We review Tuscan-2 squares found by computer, and discuss nine questions. © 1989 Academic Press, Inc.

INTRODUCTION

Our main purpose in this article is to announce the theorem which is proved in Section 1. It denies the existence of a polygonal path generating an $n \times (n+1)$ circular Tuscan-n array except when $n+1$ is prime. Our efforts in [1] had come as close as ruling out all but thirty composite values of $n+1$ below 1001. Subsequently it was discovered (by Tuvi Etzion) that [2] could supply one of the key steps in a full proof.

According to the terminology introduced in [4], and amplified in [1], an *Italian square* is an $n \times n$ array in which each of the symbols $1, 2, \ldots, n$ appears exactly once in each row. A *Tuscan-K square* is an Italian square with the further property that for any two symbols a and b, and for each t from 1 to K, there is at most one row in which b is the tth symbol to the right of a.

A *circular Tuscan-K array* is an $n \times (n+1)$ array in which each of the $n+1$ symbols $0, 1, 2, \ldots, n$ appears exactly once in each of the n rows, and in which the Tuscan-K property holds when the rows are taken to be circular.

A Tuscan square is what we now call a Tuscan-1 square and is exactly the same as a row complete row Latin square in the terminology of [3].

*This research was supported in part by the Office of Naval Research under Contract NOOO14-84-K-0189.

0196-8858/89 $7.50

A Tuscan-$(n-1)$ square is the same as a Florentine square. If it is Latin, we may call it a Vatican square.

An $n \times (n+1)$ circular Tuscan-n array is also called a circular Florentine array.

All known Florentine squares are actually Vatican and circular and can be obtained simply by writing down the multiplication table for integers mod p, where p is prime. Also they all come from polygonal path constructions where the path is given by the log function in $GF(p)$.

To describe the polygonal path construction let the vertices of a regular n-gon be numbered $1, 2, \ldots, n$ in counterclockwise order. A path which starts at one vertex, ends at another, and proceeds along directed chords to visit the n vertices once each, is what we call a *polygonal path*. If the picture of one polygonal path can be rotated to make it coincide with another, we consider them the same. Thus $(n-1)!$ is the number of different polygonal paths.

Rotation of any polygonal path constructs a Latin square as follows. Put the starting vertex of the path on the vertex of the n-gon which is numbered 1. Let the order in which the path visits the numbers 1 to n determine their order in the first row of the Latin square. Then rotating the polygonal path successively to start at $2, 3, \ldots, n$ determines the second row, third row, and so on to the nth row.

Several basic facts are proved in [1] about any $n \times n$ square L generated by the polygonal path construction. L is always an *isotope* [3] of C_n, the table for the additive group of integers modulo n. L cannot be a Tuscan-1 square when n is odd > 1. And actually an odd Tuscan square can never be obtained by row-column-symbol permutations of C_n (for odd $n > 1$). For even n, on the other hand, many L's are Tuscan squares—14,888 in standard form when $n = 14$.

Section 2 presents a detailed example showing the equivalence, proved in [1], between a singly periodic Costas array and a Florentine square obtained by the polygonal path construction.

A circular Tuscan-K array (not required to be polygonal path) leads to the construction of K pairwise orthogonal Latin squares. We give two illustrations of this theorem in Section 3.

The nine questions discussed in Section 4 serve to outline what we do not know about these various combinatorial designs.

1. The Main Result

First we quote two previous results. One is a lemma proved (by Tuvi Etzion) in [1].

LEMMA E1. *If a polygonal path $X_1, X_2, \ldots, X_n$ produces an $n \times (n+1)$ circular Tuscan-K array, then for each t from 1 to K,*

$$X_{n+1-t} \equiv \frac{n}{2} + X_t \pmod{n}.$$

In the case where $k = n$, which we use here, the conclusion of the lemma is equivalent to saying that the picture of the path is symmetric about the center of the polygon.

We shall use $0, 1, 2, \ldots, (n-1)$ to label the vertices of the n-gon and always take $X_1 = 0$. The array generated by $X_1, X_2, \ldots, X_n$ is made circular by adjoining a column full of asterisks between column n and column 1. We call this column 0 and let $X_0 = *$ which is not an integer—it will do no harm to think of $*$ as the "log of zero." The arithmetic we need has $X_i + * = * + X_i = * = X_i - * = * - X_i$ for any i from 1 to n. The subscripts $0, 1, 2, \ldots, n$ are to be taken $\bmod(n+1)$, so we may conveniently write $X_{-t} = X_{n+1-t}$.

The other previous result was proved by John B. Kelly [2] in 1954. Rephrased in our notation it gives sufficient conditions to guarantee that $n+1$ is prime.

THEOREM K. *Consider the additive group of integers $\{0, 1, 2, \ldots, n\}$ $\bmod(n+1)$. Suppose subsets A and B exist such that $A \cap B$ is empty, $A \cup B = \{1, 2, \ldots, n\}$, and $1 \in A$.*

In Case I, $n + 1 = 4m + 1$, and suppose further that I.1 and I.2 hold:

I.1. For each $a \in A$ the set $a + B$ contains m elements of A and m elements of B.

I.2. For each $b \in B$ the set $b + A$ contains m elements of B and m elements of A.

Then $n + 1$ must be prime.

In Case II, $n + 1 = 4m - 1$, and suppose further that II.1. and II.2. hold:

II.1. $|A| = |B| = 2m - 1$.

II.2. For each $a \in A$, the set $a + B$ contains 0, $m - 1$ elements of A, and $m - 1$ elements of B.

Then $n + 1$ must be prime.

We are ready to state and prove our main result.

THEOREM E. *If an $n \times (n+1)$ polygonal path circular Florentine (Tuscan-n) array exists, then $n+1$ is prime.*

Proof. Using Lemma E1 we aim to satisfy the conditions of Theorem K. We are assuming that $X_0, X_1, X_2, \dots, X_n$ has $X_0 = *$, $X_1 = 0$, and that $X_2, \dots, X_n$ is a permutation of the integers 1 to $n - 1$, taken mod n.

Let $A = \{i\colon X_i \text{ is even}\}$

Let $B = \{i\colon X_i \text{ is odd}\}$.

With the subscripts taken $\bmod\, n + 1$ it is immediate that $A \cap B$ is empty, $A \cup B = \{1, 2, \dots, n\}$, and $1 \in A$. Also since n is even, $|A| = |B| = n/2$.

If we take any subscript $t \in A \cup B$, it will partition $X_0, X_1, \dots, X_n$ into cycles of the form

$$\dots, X_{i-t}, X_i, X_{i+t}, X_{i+2t}, \dots .$$

Furthermore, because the polygonal path generates a circular Tuscan-n array the successive differences in cycles, $X_{i+t} - X_i$, will take each of the values $1, 2, \dots, n - 1$ exactly once, and take $* = X_t - X_0 = X_0 - X_{-t}$ twice. Thus in both cases below there will be $n/2$ odd differences and $n/2 - 1$ even differences.

One more general fact will be taken for granted. In any cycle of ODDs and EVENs, the number of times ODD is followed by EVEN necessarily equals the number of times EVEN is followed by ODD.

Case I. $(n + 1 = 4m + 1)$. In this case $n/2$ is even so we know by Lemma E1 that $X_t \equiv X_{-t} \pmod 2$. This tells us that in the cycle which contains $X_0 = *$ (as well as in all the other cycles if there are any) it will occur the same number of times that X_i is odd and X_{i+t} is even as that X_i is even and X_{i+t} is odd. Consequently there are m subscripts i such that $i \in A$ and $i + t \in B$, as well as m subscripts i such that $i \in B$ and $i + t \in A$, since the number of odd differences all together is $2m = n/2$. Where the $2m - 1$ even differences go will depend on t.

I.1. If $t \in A$, then X_t is even, so X_{-t} is even, whereas $X_{-t+t} = X_0$ is not even. This guarantees that only $m - 1$ of the remaining subscripts $i \in A$ are such that $i + t \in A$. Thus we infer that there are exactly m subscripts $i \in B$ such that $i + t \in B$. In other words, condition I.1. of Theorem K is satisfied.

I.2. If $t \in B$, then X_t is odd, so X_{-t} is odd, $X_{-t+t} = X_0$ is not odd, and only $m - 1$ of the remaining subscripts $i \in B$ are such that $i + t \in B$. Thus we infer that there are m subscripts $i \in A$ such that $i + t \in A$. In other words, condition I.2. of Theorem K is satisfied.

We have found that $n + 1$ is prime in Case I.

Case II. $(n+1=4m-1)$. In this case $n/2$ is odd so we know by Lemma E that $X_t \not\equiv X_{-t} \pmod 2$.

II.1. We already know that $|A| = |B| = 2m-1$.

II.2. If $t \in A$, then X_t is even, so X_{-t} is odd, $X_{-t+t} = X_0$, $-t \in B$, and $0 \in t + B$. From this we can also see that $m-1$ is the number of times that X_i is odd and X_{i+t} is even, while m is the number of times that X_i is even and X_{i+t} is odd. The $2m-2 = n/2-1$ even differences can only be accounted for by having $m-1$ times that X_i is even and X_{i+t} is even, beside $m-1$ times that X_i is odd and X_{i+t} is odd. All together there is one subscript $i \in B$ such that $i+t=0$, $m-1$ subscripts $i \in B$ such that $i+t \in A$, and $m-1$ subscripts $i \in B$ such that $i+t \in B$. In other words, condition II.2. of Theorem K is satisfied.

We have found that $n+1$ is prime in Case II. That completes the proof of Theorem E. □

2. A Singly Periodic Costas Array Is Equivalent to a Polygonal Path Vatican Square

Let $X_1, X_2, \ldots, X_n$ be a permutation of the integers $0, 1, 2, \ldots, (n-1)$. Suppose that for each fixed t from 1 to $(n-1)$, and for $1 \le i < j \le n-t$ the tth differences are all distinct mod n, that is $X_{i+t} - X_i \not\equiv X_{j+t} - X_j \pmod n$.

If such a sequence exists it provides the essential pattern for an $\infty \times n$ singly periodic Costas array, or equally well a polygonal path which produces an $n \times n$ Vatican square.

Such a sequence is only known to exist when $n+1$ is prime, α is primitive, and taking logs to the base α in $GF(n+1)$, we let $X_i = \log i$. To illustrate, let $n+1 = 11$, $\alpha = 2$:

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
0	1	8	2	4	9	7	3	6	5

In each row of the difference triangle in Fig. 1, the differences are distinct mod n. Figure 2 exhibits the $\infty \times 10$ singly periodic Costas array produced by the sequence 0182497365, Fig. 3, the polygonal path, and Fig. 4, the Vatican square generated by the polygonal path $X_1, \ldots, X_{10}$ where $\alpha = 2$.

The question "Does a singly periodic $\infty \times n$ Costas array exist for any composite $n+1$?" was listed in [5], with mention that the answer is NO if $n+1$ is even, and NO for $n \le 16$. As reported in [1] we now know the answer is NO for $n \le 20$. Also when $n+1$ is prime the question remains

$$\begin{array}{ccccccccccccccccccc}
\underline{0} & & \underline{1} & & \underline{8} & & \underline{2} & & \underline{4} & & \underline{9} & & \underline{7} & & \underline{3} & & \underline{6} & & \underline{5} \\
& 1 & & 7 & & -6 & & 2 & & 5 & & -2 & & -4 & & 3 & & -1 & \\
& & 8 & & 1 & & -4 & & 7 & & 3 & & -6 & & -1 & & 2 & & \\
& & & 2 & & 3 & & 1 & & 5 & & -1 & & -3 & & -2 & & & \\
& & & & 4 & & 8 & & -1 & & 1 & & 2 & & -4 & & & & \\
& & & & & 9 & & 6 & & -5 & & 4 & & 1 & & & & & \\
& & & & & & 7 & & 2 & & -2 & & 3 & & & & & & \\
& & & & & & & 3 & & 5 & & -3 & & & & & & & \\
& & & & & & & & 6 & & 4 & & & & & & & & \\
& & & & & & & & & 5 & & & & & & & & &
\end{array}$$

FIG. 1. The difference triangle for $X_i = \log_2 i$ over $GF(11)$.

	1	2	3	4	5	6	7	8	9	10
3								•		
2				•						
1		•								
0	•									
9						•				
8			•							
7							•			
6									•	
5										•
4					•					
3								•		
2				•						
1		•								
0	•									
9						•				
8			•							
7							•			
6									•	
5										•
4					•					
3								•		
2				•						
1		•								
0	•									
9						•				
8			•							

FIG. 2. The $\infty \times 10$ singly periodic Costas array produced by the sequence 0182497365.

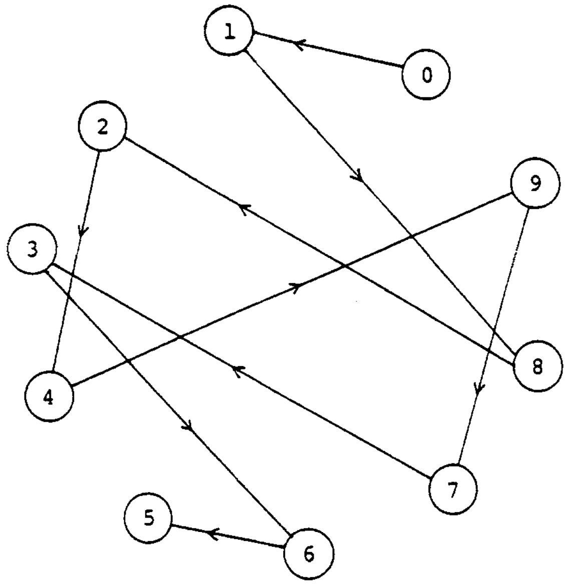

FIG. 3. The polygonal path.

whether any examples exist other than by the log construction just illustrated. (The Welch construction, as it is called [5] in the context of Costas arrays, is not different.)

One more equivalent form of the problem is illustrated in Fig. 5. An $n \times n$ polygonal path Tuscan-$(n-1)$ square exists if and only if it is possible to so permute the rows and columns of the addition table for the cyclic group of integers mod n that the resulting matrix $C(i, j)$ has $C(i, j) = 0$ when $i = j$, and $C(i, j) \neq C(i+k, j+k)$, where $i \neq j$, and

0	1	8	2	4	9	7	3	6	5
1	2	9	3	5	0	8	4	7	6
2	3	0	4	6	1	9	5	8	7
3	4	1	5	7	2	0	6	9	8
4	5	2	6	8	3	1	7	0	9
5	6	3	7	9	4	2	8	1	0
6	7	4	8	0	5	3	9	2	1
7	8	5	9	1	6	4	0	3	2
8	9	6	0	2	7	5	1	4	3
9	0	7	1	3	8	6	2	5	4

FIG. 4. The Vatican square generated by the polygonal path $X_1, \ldots, X_{10}$ where $\alpha = 2$, $X_i = \log i$ over $GF(11)$.

0	9	2	8	6	1	3	7	4	5
1	0	3	9	7	2	4	8	5	6
8	7	0	6	4	9	1	5	2	3
2	1	4	0	8	3	5	9	6	7
4	3	6	2	0	5	7	1	8	9
9	8	1	7	5	0	2	6	3	4
7	6	9	5	3	8	0	4	1	2
3	2	5	1	9	4	6	0	7	8
6	5	8	4	2	7	9	3	0	1
5	4	7	3	1	6	8	2	9	0

FIGURE 5

$1 \le i < i + k \le n$, and $1 \le j < j + k \le n$.

3. K Pairwise Orthogonal Latin Squares from a Circular Tuscan-K Array

THEOREM A. *If an $n \times (n+1)$ circular Tuscan-K array exists, then K orthogonal $(n+1) \times (n+1)$ Latin squares exist.*

The full proof is given in [1] that the following construction obtains a set $L_1, \ldots, L_K$ of pairwise orthogonal Latin squares, when an $n \times (n+1)$ circular Tuscan-K array A is given.

We may take it that each circular row of A is a permutation of the symbols $0, 1, 2, \ldots, n$. We name the rows of A row1, row2, …, row$n$. We name the columns of A modulo $n+1$ col.0, col.1, …, col.n so that, for example, we consider the symbol in row r, column 1 of A to be two steps to the right of a symbol in row r, column n of A.

Using "$L_t(i, j)$" to denote the symbol in row i, column j of L_t, we describe first L_1, and then $L_2, \ldots, L_K$ as follows:

$$L_1(i, j) = \begin{cases} 0 & \text{if } i = j \\ r & \text{if the symbol } j \text{ occurs one step to the} \\ & \text{right of symbol } i \text{ in row } r \text{ of } A. \end{cases}$$

For each t from 2 to K,

$$L_t(i, j) = \begin{cases} j & \text{if } i = j \\ h & \text{if the symbol } h \text{ is } t \text{ steps to the right of the} \\ & \text{symbol } i \text{ in the row of } A \text{ in which the symbol} \\ & j \text{ is one step to the right of } i. \end{cases}$$

a

0	1	2	3	4
0	2	4	1	3
0	3	1	4	2
0	4	3	2	1

b

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

c

0	2	4	1	3
4	1	3	0	2
3	0	2	4	1
2	4	1	3	0
1	3	0	2	4

d

0	3	1	4	2
3	1	4	2	0
1	4	2	0	3
4	2	0	3	1
2	0	3	1	4

e

0	4	3	2	1
2	1	0	4	3
4	3	2	1	0
1	0	4	3	2
3	2	1	0	4

FIG. 6. (a) The array A; (b) L_1 from A; (c) L_2 from A; (d) L_3 from A; (e) L_4 from A.

The construction is illustrated in Fig. 6 with a 4×5 circular Tuscan-4 array A.

The 14×15 circular Tuscan-2 array B, shown in Fig. 7 along with the orthogonal Latin squares L_1 and L_2 derived from it, is also an example of a non-Latin Tuscan-2 square obtained by the add-zero transformation [4] from a polygonal path construction.

Neither L_1 nor L_2 is a group table isotope since they both contain violations of the quadrangle criterion [3].

4. Nine Questions

Q1. For odd $n > 7$, do any circular $n \times (n + 1)$ Tuscan-2 arrays exist? If any do exist, the add-zero transformation would guarantee some odd non-Latin Tuscan-2 squares.

Q2. For odd $n > 7$, do any $n \times n$ Tuscan-2 squares exist? We conjecture YES, although none are known.

Q3. For odd $n > 1$, do any $n \times (n + 1)$ circular Tuscan-n arrays exist? The answer is NO. This is proved in [1].

Q4. For odd $n > 1$, do any $n \times n$ Florentine squares exist? We conjecture NO.

Q5. For all even $n > 8$, do circular $n \times (n + 1)$ Tuscan-2 arrays exist? We conjecture YES. Although there are none for $n = 8$, there are examples for all even $n > 8$ up to $n = 50$. The computer search reported in [1] was

a

															Row Nos.
0	2	13	5	1	10	11	4	3	8	12	6	9	7	14	1
0	6	2	11	12	5	4	9	13	7	10	8	14	1	3	2
0	8	11	9	14	2	4	1	7	3	12	13	6	5	10	3
0	7	6	11	1	9	12	10	14	3	5	2	8	4	13	4
0	1	8	7	12	2	10	13	11	14	4	6	3	9	5	5
0	12	14	5	7	4	10	6	1	2	9	8	13	3	11	6
0	4	12	1	13	14	6	8	5	11	7	2	3	10	9	7
0	14	7	9	6	12	8	3	4	11	10	1	5	13	2	8
0	3	1	14	8	10	7	13	9	4	5	12	11	2	6	9
0	10	5	6	13	12	3	7	1	4	2	14	9	11	8	10
0	13	4	8	2	5	3	14	10	12	9	1	11	6	7	11
0	5	9	3	6	4	14	11	13	10	2	12	7	8	1	12
0	11	3	13	8	9	2	1	6	10	4	7	5	14	12	13
0	9	10	3	2	7	11	5	8	6	14	13	1	12	4	14

b

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	5	1	9	7	12	2	4	3	14	10	13	6	11	8
1	12	0	6	2	10	8	13	3	5	4	1	11	14	7	9
2	8	13	0	7	3	11	9	14	4	6	5	2	12	1	10
3	2	9	14	0	8	4	12	10	1	5	7	6	3	13	11
4	14	3	10	1	0	9	5	13	11	2	6	8	7	4	12
5	5	1	4	11	2	0	10	6	14	12	3	7	9	8	13
6	9	6	2	5	12	3	0	11	7	1	13	4	8	10	14
7	11	10	7	3	6	13	4	0	12	8	2	14	5	9	1
8	10	12	11	8	4	7	14	5	0	13	9	3	1	6	2
9	7	11	13	12	9	5	8	1	6	0	14	10	4	2	3
10	3	8	12	14	13	10	6	9	2	7	0	1	11	5	4
11	6	4	9	13	1	14	11	7	10	3	8	0	2	12	5
12	13	7	5	10	14	2	1	12	8	11	4	9	0	3	6
13	4	14	8	6	11	1	3	2	13	9	12	5	10	0	7
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0

c

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	**0**	8	13	1	12	9	2	6	11	10	5	3	14	4	7
1	5	**1**	9	0	2	13	10	3	7	12	11	6	4	14	8
2	14	6	**2**	10	1	3	0	11	4	8	13	12	7	5	9
3	6	14	7	**3**	11	2	4	1	12	5	9	0	13	8	10
4	9	7	14	8	**4**	12	3	5	2	13	6	10	1	0	11
5	1	10	8	14	9	**5**	13	4	6	3	0	7	11	2	12
6	3	2	11	9	14	10	**6**	0	5	7	4	1	8	12	13
7	13	4	3	12	10	14	11	**7**	1	6	8	5	2	9	0
8	10	0	5	4	13	11	14	12	**8**	2	7	9	6	3	1
9	4	11	1	6	5	0	12	14	13	**9**	3	8	10	7	2
10	8	5	12	2	7	6	1	13	14	0	**10**	4	9	11	3
11	12	9	6	13	3	8	7	2	0	14	1	**11**	5	10	4
12	11	13	10	7	0	4	9	8	3	1	14	2	**12**	6	5
13	7	12	0	11	8	1	5	10	9	4	2	14	3	**13**	6
14	2	3	4	5	6	7	8	9	10	11	12	13	0	1	**14**

FIG. 7. (a) The array B; (b) L_1 from B; (c) L_2 from B.

able to go that far because the candidates were restricted to symmetric polygonal path constructions.

Q6. For all even n, do $n \times n$ Tuscan-2 squares exist? We conjecture YES. However, we have not proven it for infinitely many values of n such that $n + 1$ is composite.

Q7. For even n, do any $n \times (n + 1)$ circular Tuscan-n arrays exist when $n + 1$ is not prime? We conjecture NO. Using Theorem A the answer is NO when the Bruck–Ryser theorem precludes a projective plane of order $n + 1$. Also Theorem E says none can come from the polygonal path construction. The first open case is $n = 14$.

Q8. For even n, do any $n \times n$ Florentine squares exist when $n + 1$ is not prime? We conjecture YES, although none are known. The first open case is $n = 14$.

Q9. Does there exist an $n \times n$ Tuscan-3 square for any n such that $n + 1$ is not prime? We conjecture YES.

References

1. S. W. Golomb, T. Etzion, and H. Taylor, Polygonal path constructions for Tuscan-K squares, *Ars Combin.*, in press.
2. J. B. Kelly, A characteristic property of quadratic residues, *Proc. Amer. Math. Soc.* **5** (1954), 36.
3. J. Dénes and A. D. Keedwell, "Latin Squares and Their Applications", Academic Press, New York/London, 1974.
4. S. W. Golomb and H. Taylor, Tuscan squares—A new family of combinatorial designs, *Ars Combin.* **20-B** (1985), 115–132.
5. S. W. Golomb and H. Taylor, Constructions and properties of Costas arrays, *Proc. IEEE* **72**, No. 9 (1984), 1143–1163.